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Representation of Lie algebra $\mathcal{T}_3$ and 2-variable 2-parameter Bessel functions

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Abstract

In this paper, we deal with the problem of framing 2-variable 2-parameter Bessel functions (2V2PBF) into the context of the representation $Q(w, m_0)$ of the 3-dimensional Lie algebra $\mathcal{T}_3$. Further, we derive generating relations and identities involving 2V2PBF. Furthermore, we deduce some new and known relations involving various other forms of generalized Bessel functions (GBF) and Bessel functions (BF) as applications of these results.

Keywords: Generalized Bessel functions; Lie group; Lie algebra; Representation theory; Generating relations

1. Introduction

Theory of special functions plays an important role in the formalism of mathematical physics. Bessel functions (BF) [17], are among the most important special functions, with very diverse applications to physics, engineering and mathematical physics ranging from abstract number theory and theoretical astronomy to concrete problems of physics and engineering.

Dattoli and his co-workers introduced and discussed various generalizations of BF within purely mathematical and applicative contexts (see for example [2–8]). Generalized Bessel functions (GBF) have become a powerful tool to investigate the dynamical aspects of physical

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problems such as electron scattering by an intense linearly polarized laser wave, multi-photon processes and undulator radiation. The analytical and numerical study of GBF has revealed their interesting properties, which in some sense can be regarded as an extension of the properties of BF to a 2-dimensional domain. In this connection, the relevance of GBF and their multi-variable extension in mathematical physics has been emphasized, since they provide analytical solutions to partial differential equations such as the multi-dimensional diffusion equation, the Schrödinger and Klein–Gordon equations.

A useful complement to the theory of GBF is offered by the introduction of 2-variable 2-parameter Bessel functions (2V2PBF) $J_n(x; z/y; z')$ defined as [8, p. 160, (4.1)]

$$J_n(x; z/y; z') = \sum_{l=-\infty}^{\infty} J_{n-2l}(x; z) J_l(y; z'), \tag{1.1}$$

where z and z' are arbitrary complex parameters.

The generating function for 2V2PBF is given as [8, p. 161, (4.41)]

$$\sum_{n=-\infty}^{\infty} J_n(x; z/y; z') t^n = \exp\left\{\frac{x}{2}\left(t - \frac{z}{t}\right) + \frac{y}{2}\left(t^2 - \frac{z'}{t^2}\right)\right\}. \tag{1.2}$$

This is the most convenient form to study the modified forms of the first-kind cylindrical GBF.

The theory of special functions from the group theoretic point of view is a well established topic, providing a unifying formalism to deal with the immense aggregate of the special functions and a collection of formulas such as the relevant differential equations, integral representations, recurrence formulae, composition theorems, etc., see for example [19,20]. Recently, some contributions related to Lie-theoretical representations of generalized Laguerre and Hermite polynomials of two variables have been given, see for example Khan [12,13], Khan and Pathan [14] and Khan and Yasmin [15].

Within the group-theoretic context, indeed a given class of special functions appears as a set of matrix elements of irreducible representations of a given Lie group. The algebraic properties of the group are then reflected in the functional and differential equations satisfied by a given family of special functions, whilst the geometry of the homogeneous space determines the nature of the integral representation associated with the family.

The Bessel functions of integral order have been shown to be connected with the faithful irreducible unitary representations of the real Euclidean group E_3 in the plane [18,22]. The Euclidean group E_3 is a real 3-parameter global Lie group, whose Lie algebra $\mathcal{E}_3$ has basis elements

$$\mathcal{J}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{1.3}$$

with commutation relations

$$[\mathcal{J}_1, \mathcal{J}_2] = 0, \qquad [\mathcal{J}_3, \mathcal{J}_1] = \mathcal{J}_2, \qquad [\mathcal{J}_3, \mathcal{J}_2] = -\mathcal{J}_1. \tag{1.4}$$

The 3-dimensional complex local Lie group T_3 is the set of all 4×4 matrices of the form

$$g = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & e^{-a} & 0 & c \\ 0 & 0 & e^{a} & b \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b, c \in \mathbb{C}. \tag{1.5}$$

A basis for the Lie algebra $\mathcal{T}_3 = L(T_3)$ is provided by the matrices

$$\mathcal{J}^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathcal{J}^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{1.6}$$

with commutation relations

$$[\mathcal{J}^3, \mathcal{J}^+] = \mathcal{J}^+, \qquad [\mathcal{J}^3, \mathcal{J}^-] = -\mathcal{J}^-, \qquad [\mathcal{J}^+, \mathcal{J}^-] = 0. \tag{1.7}$$

Further, we observe that the complex matrices

$$\mathcal{J}^{+\prime} = -\mathcal{J}_2 + i\mathcal{J}_1, \quad \mathcal{J}^{-\prime} = \mathcal{J}_2 + i\mathcal{J}_1, \quad \mathcal{J}^{3\prime} = i\mathcal{J}_3, \quad i = \sqrt{-1}, \tag{1.8}$$

satisfy the commutation relations identical with (1.7). Thus we say that $\mathcal{T}_3$ is the *comlexification* of $\mathcal{E}_3$ and $\mathcal{E}_3$ is a *real form* of $\mathcal{T}_3$ [10]. Due to this relationship between $\mathcal{T}_3$ and $\mathcal{E}_3$, the abstract irreducible representation $Q(w, m_0)$ of $\mathcal{T}_3$ [16] induces an irreducible representation of $\mathcal{E}_3$.

In this paper, we deal with the problem of framing 2V2PBF into the context of the representation $Q(w, m_0)$ of the Lie algebra $\mathcal{T}_3$. In Section 2, we give a review of the basic properties of 2V2PBF $J_n(x; z/y; z')$ and their connections with other forms of GBF. In Section 3, we derive generating relations involving 2V2PBF. In Section 4, we obtain many new relations involving various other forms of GBF, also we mention some known relations. Finally, in Section 5, concluding remarks are given.

2. Properties and special cases of 2V2PBF $J_n(x; z/y; z')$

The 2V2PBF $J_n(x; z/y; z')$ defined by Eqs. (1.1) and (1.2) satisfy the following differential and pure recurrence relations:

$$\begin{aligned} \frac{\partial}{\partial x} J_n(x; z/y; z') &= \frac{1}{2}\big(J_{n-1}(x; z/y; z') - z J_{n+1}(x; z/y; z')\big), \\ \frac{\partial}{\partial y} J_n(x; z/y; z') &= \frac{1}{2}\big(J_{n-2}(x; z/y; z') - z' J_{n+2}(x; z/y; z')\big), \\ \frac{\partial}{\partial z} J_n(x; z/y; z') &= -\frac{x}{2} J_{n+1}(x; z/y; z'), \\ \frac{\partial}{\partial z'} J_n(x; z/y; z') &= -\frac{y}{2} J_{n+2}(x; z/y; z') \end{aligned} \tag{2.1}$$

and

$$\begin{aligned} n J_n(x; z/y; z') = {} & \frac{x}{2}\big(J_{n-1}(x; z/y; z') + z J_{n+1}(x; z/y; z')\big) \\ & + y\big(J_{n-2}(x; z/y; z') + z' J_{n+2}(x; z/y; z')\big). \end{aligned} \tag{2.2}$$

The differential equation satisfied by 2V2PBF $J_n(x; z/y; z')$ is

$$\begin{aligned} \bigg(& -\frac{1}{z}\frac{\partial^2}{\partial x^2} + \frac{4y^2}{x^2 z}\frac{\partial^2}{\partial y^2} + \frac{16 z'^2}{x^2 z}\frac{\partial^2}{\partial z'^2} - \frac{16 y z'}{x^2 z}\frac{\partial^2}{\partial y \partial z'} - \frac{1}{xz}\frac{\partial}{\partial x} - \frac{4(n-1)y}{x^2 z}\frac{\partial}{\partial y} \\ & + \frac{8(n+2)z'}{x^2 z}\frac{\partial}{\partial z'} + \frac{n^2}{x^2 z} - 1 \bigg) J_n(x; z/y; z') = 0. \end{aligned} \tag{2.3}$$

It is also worth to mention the following expansions of $J_n(x;z/y;z')$ in terms of 2-variable Hermite Kampé de Fériet polynomials (2VHKdFP) $H_n(x,y)$ [8]:

$$J_n(x;z/y;z') = \sum_{r=0}^{\infty} \frac{H_{n+r}(\frac{x}{2},\frac{y}{2})H_r(-\frac{x}{2}z,-\frac{y}{2}z')}{(n+r)!r!}, \quad n \geqslant 0, \tag{2.4}$$

and

$$J_n(x;z/y;z') = \sum_{r=0}^{\infty} \frac{J_{n+r}(x;\xi/y;\eta)H_r(\frac{x}{2}(\xi-z),\frac{y}{2}(\eta-z'))}{r!}, \tag{2.5}$$

where ξ and η are arbitrary parameters and 2VHKdFP $H_n(x,y)$ are defined by the generating function [8, p. 151, (3.1)]

$$\sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!} = \exp\bigl(xt+yt^2\bigr). \tag{2.6}$$

Next, we note the following special cases of 2V2PBF $J_n(x;z/y;z')$:

(1) $$J_n(x;1/y;1) = J_n(x,y), \tag{2.7}$$

where $J_n(x,y)$ denotes 2-variable Bessel functions (2VBF) defined by generating function [7, p. 24, (1.8(a))]

$$\sum_{n=-\infty}^{\infty} J_n(x,y)t^n = \exp\left\{\frac{x}{2}\left(t-\frac{1}{t}\right)+\frac{y}{2}\left(t^2-\frac{1}{t^2}\right)\right\}. \tag{2.8}$$

(2) $$J_n(x;-1/y;-1) = I_n(x,y), \tag{2.9}$$

where $I_n(x,y)$ denotes 2-variable modified Bessel functions (2VMBF) defined by the generating function [4, p. 331, (2.11a)]

$$\sum_{n=-\infty}^{\infty} I_n(x,y)t^n = \exp\left\{\frac{x}{2}\left(t+\frac{1}{t}\right)+\frac{y}{2}\left(t^2+\frac{1}{t^2}\right)\right\}. \tag{2.10}$$

(3) $$J_n(x;0/y;z') = \sum_{s=0}^{[\frac{n}{2}]} \frac{(\frac{x}{2})^{n-2s}}{(n-2s)!} J_s(y;z'), \tag{2.11}$$

where $J_n(y;z')$ denotes 1-variable 1-parameter Bessel functions (1V1PBF) [8, p. 162].

Similarly,

$$J_n(x;z/y;0) = \sum_{s=0}^{\infty} \frac{(\frac{y}{2})^s}{s!} J_{n-2s}(x;z). \tag{2.12}$$

(4) $$J_n(x;0/y;0) = \frac{1}{n!} H_n\left(\frac{x}{2},\frac{y}{2}\right), \tag{2.13}$$

where $H_n(x,y)$ is given by Eq. (2.6).

(5) $$J_n(x;i/y;i)=\exp\left(-\frac{in\pi}{4}\right)J_n\left(x\exp\left(\frac{\pi i}{4}\right),y\exp\left(\frac{\pi i}{4}\right),\frac{\pi i}{4}\right), \tag{2.14}$$

which can be viewed as a kind of generalized Kelvin function [8, p. 164]. Further the function

$$J_n(x;i/y;1)=\exp\left(-\frac{in\pi}{4}\right)J_n\left(x\exp\left(\frac{\pi i}{4}\right),y;i\right), \tag{2.15}$$

provides another interesting form [8, p. 164].

(6) $$J_n(x;1/0;z')=J_n(x), \tag{2.16}$$

where $J_n(x)$ denotes BF defined by the generating function [17, p. 113, (4)]

$$\sum_{n=-\infty}^{\infty} J_n(x)t^n=\exp\left(\frac{x}{2}\left(t-\frac{1}{t}\right)\right). \tag{2.17}$$

3. Representation $Q(\omega,m_0)$ of $\mathcal{T}_3$ and generating relations

Miller [16] have determined realizations of the irreducible representation $Q(\omega,m_0)$ of $\mathcal{T}_3$ where $\omega,m_0\in\mathbb{C}$ such that $\omega\neq 0$ and $0\leqslant \operatorname{Re} m_0<1$. The spectrum S of this representation is the set $\{m_0+k\colon k \text{ an integer}\}$, and the representation space V has a basis $(f_m)_{m\in S}$, such that

$$\begin{aligned}&J^3 f_m=mf_m,\qquad J^+f_m=\omega f_{m+1},\qquad J^-f_m=\omega f_{m-1},\\&C_{0,0}f_m=\left(J^+J^-\right)f_m=\omega^2 f_m,\quad \omega\neq 0.\end{aligned} \tag{3.1}$$

The commutation relations satisfied by the operators J^3, $J^\pm$ are

$$\left[J^3,J^+\right]=J^+,\qquad \left[J^3,J^-\right]=-J^-,\qquad \left[J^+,J^-\right]=0. \tag{3.2}$$

In order to find the realizations of this representation on spaces of functions of two complex variables x and y, Miller [16, pp. 59–60] has taken the functions $f_m(x,y)=Z_m(x)e^{my}$, such that relations (3.1) are satisfied for all $m\in S$, where the differential operators J^3, $J^\pm$ are given by

$$\begin{aligned}&J^3=\frac{\partial}{\partial y},\\&J^+=e^y\left[\frac{\partial}{\partial x}-\frac{1}{x}\frac{\partial}{\partial y}\right],\\&J^-=e^{-y}\left[-\frac{\partial}{\partial x}-\frac{1}{x}\frac{\partial}{\partial y}\right].\end{aligned} \tag{3.3}$$

In particular, we are looking for the functions $f_m(x,y,t;z,z')=Z_m(x,y;z,z')t^m$ such that relations (3.1) are satisfied for all $m\in S$.

We take the set of linear differential operators $J^{3\prime}$, $J^{\pm\prime}$ as follows:

$$\begin{aligned}&J^{3\prime}=t\frac{\partial}{\partial t},\\&J^{+\prime}=\frac{t}{z}\frac{\partial}{\partial x}+\frac{2yt}{xz}\frac{\partial}{\partial y}-\frac{4z't}{xz}\frac{\partial}{\partial z'}-\frac{t^2}{xz}\frac{\partial}{\partial t},\\&J^{-\prime}=-\frac{1}{t}\frac{\partial}{\partial x}+\frac{2y}{xt}\frac{\partial}{\partial y}-\frac{4z'}{xt}\frac{\partial}{\partial z'}-\frac{1}{x}\frac{\partial}{\partial t},\end{aligned} \tag{3.4}$$

and note that these operators satisfy the commutation relations identical to (3.2).

In terms of the functions $Z_m(x,y;z,z')$ and using operators (3.4), relations (3.1) reduce to

$$\begin{aligned}
&\text{(i)} \quad \left[\frac{1}{z}\frac{\partial}{\partial x}+\frac{2y}{xz}\frac{\partial}{\partial y}-\frac{4z'}{xz}\frac{\partial}{\partial z'}-\frac{m}{xz}\right]Z_m(x,y;z,z')=\omega Z_{m+1}(x,y;z,z'),\\
&\text{(ii)} \quad \left[-\frac{\partial}{\partial x}+\frac{2y}{x}\frac{\partial}{\partial y}-\frac{4z'}{x}\frac{\partial}{\partial z'}-\frac{m}{x}\right]Z_m(x,y;z,z')=\omega Z_{m-1}(x,y;z,z'),\\
&\text{(iii)} \quad \left[-\frac{1}{z}\frac{\partial^2}{\partial x^2}+\frac{4y^2}{x^2z}\frac{\partial^2}{\partial y^2}+\frac{16z'^2}{x^2z}\frac{\partial^2}{\partial z'^2}-\frac{16yz'}{x^2z}\frac{\partial^2}{\partial y\partial z'}-\frac{1}{xz}\frac{\partial}{\partial x}-\frac{4(m-1)y}{x^2z}\frac{\partial}{\partial y}\right.\\
&\qquad\left.+\frac{8(m+2)z'}{x^2z}\frac{\partial}{\partial z'}+\frac{m^2}{x^2z}\right]Z_m(x,y;z,z')=\omega^2 Z_m(x,y;z,z'). \qquad (3.5)
\end{aligned}$$

The complex constant ω in these equations and in relations (3.1) is clearly nonessential. Hence, without any loss of generality we can assume $\omega=-1$. For this choice of ω, and in terms of the functions $Z_m(x)$, relations (3.1) become [16, p. 60, (3.25)]

$$\begin{aligned}
&\text{(i)} \quad \left[\frac{d}{dx}-\frac{m}{x}\right]Z_m(x)=-Z_{m+1}(x),\\
&\text{(ii)} \quad \left[\frac{d}{dx}+\frac{m}{x}\right]Z_m(x)=Z_{m-1}(x),\\
&\text{(iii)} \quad \left[-\frac{d^2}{dx^2}-\frac{1}{x}\frac{d}{dx}+\frac{m^2}{x^2}\right]Z_m(x)=Z_m(x). \qquad (3.6)
\end{aligned}$$

We observe that (i) and (ii) of Eqs. (3.6) agree with the conventional recursion relations for BF $J_m(x)$ and (iii) coincides with the differential equation for $J_m(x)$. Thus we see that $Z_m(x)=J_m(x)$ is a solution of Eqs. (3.6) for all $m\in S$.

Similarly, we see that for $\omega=-1$, (iii) of Eqs. (3.5) coincides with the differential equation (2.3) of 2V2PBF $J_m(x;z/y;z')$. In fact, for all $m\in S$ the choice for $Z_m(x,y;z,z')=J_m(x;z/y;z')$ satisfy Eqs. (3.5). It follows from the above discussion that the functions $f_m(x,y,t;z,z')=J_m(x;z/y;z')t^m$, $m\in S$, form a basis for a realization of the representation $Q(-1,m_0)$ of $\mathcal{T}_3$. By using [16, p. 18, Theorem 1.10], this representation of $\mathcal{T}_3$ can be extended to a local multiplier representation T [16, p. 17] of T_3 defined on $\mathcal{F}$, the space of all functions analytic in a neighbourhood of the point $(x^0,y^0,t^0;z^0,z'^0)=(1,0,1,1,1)$. Using operators (3.4), the local multiplier representation takes the form

$$\begin{aligned}
&\left[T\left(\exp a\mathcal{J}^3\right)f\right](x,y,t;z,z')=f\left(x,y,e^a t;z,z'\right),\\
&\left[T\left(\exp b\mathcal{J}^+\right)f\right](x,y,t;z,z')\\
&\quad=f\left(x\left(1+\frac{2bt}{xz}\right)^{1/2},y\left(1+\frac{2bt}{xz}\right),t\left(1+\frac{2bt}{xz}\right)^{-1/2};z,z'\left(1+\frac{2bt}{xz}\right)^{-2}\right),\\
&\left[T\left(\exp c\mathcal{J}^-\right)f\right](x,y,t;z,z')\\
&\quad=f\left(x\left(1-\frac{2c}{xt}\right)^{1/2},y\left(1-\frac{2c}{xt}\right)^{-1},t\left(1-\frac{2c}{xt}\right)^{1/2};z,z'\left(1-\frac{2c}{xt}\right)^{2}\right),\\
&\left|\frac{2bt}{xz}\right|<1,\ \left|\frac{2c}{xt}\right|<1, \qquad (3.7)
\end{aligned}$$

for $f \in \mathcal{F}$. If $g \in T_3$ is given by Eq. (1.5), then

$$g = (\exp b\mathcal{J}^+)(\exp c\mathcal{J}^-)(\exp a\mathcal{J}^3).$$

Therefore, for $f \in \mathcal{F}$ and g in a sufficiently small neighbourhood of the identity we have

$$[T(g)f](x, y, t; z, z') = [T(\exp b\mathcal{J}^+)T(\exp c\mathcal{J}^-)T(\exp a\mathcal{J}^3)f](x, y, t; z, z'),$$

and hence we obtain

$$[T(g)f](x, y, t; z, z') = f\left(x(\phi\psi)^{1/2}, y\phi\psi^{-1}, e^a t\phi^{-1/2}\psi^{1/2}; z, z'\phi^{-2}\psi^2\right),$$

where

$$\phi := 1 + \frac{2bt}{xz}, \qquad \psi := 1 - \frac{2c}{xt} \quad \text{and} \quad \left|\frac{2bt}{xz}\right| < 1, \qquad \left|\frac{2c}{xt}\right| < 1. \tag{3.8}$$

The matrix elements of $T(g)$ with respect to the analytic basis $(f_m)_{m\in S}$ are the functions $A_{lk}(g)$ uniquely determined by $Q(-1, m_0)$ of $\mathcal{T}_3$, and we obtain relations

$$[T(g)f_{m_0+k}](x, y, t; z, z') = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l}(x, y, t; z, z'), \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots, \tag{3.9}$$

which simplifies to the identity

$$\begin{aligned} &e^{ma}\phi^{-m/2}\psi^{m/2} J_m\left(x(\phi\psi)^{1/2}; z/y\phi\psi^{-1}; z'\phi^{-2}\psi^2\right) \\ &\quad = \sum_{l=-\infty}^{\infty} A_{l,m-m_0}(g) J_{m_0+l}(x; z/y; z') t^{m_0+l-m}, \end{aligned} \tag{3.10}$$

and the matrix elements $A_{lk}(g)$ are given by [16, p. 56, (3.12)′],

$$A_{lk}(g) = \frac{e^{(m_0+k)a}(-1)^{|k-l|}(b)^{(l-k+|k-l|)/2}(c)^{(k-l+|k-l|)/2}}{|k-l|!}\,{}_0F_1[-; |k-l|+1; bc], \tag{3.11}$$

for all integral values of l, k and where ${}_0F_1$ denotes confluent hypergeometric function [17]. Substituting (3.11) into (3.10), we obtain the generating relation

$$\begin{aligned} &\phi^{-m/2}\psi^{m/2} J_m\left(x(\phi\psi)^{1/2}; z/y\phi\psi^{-1}; z'\phi^{-2}\psi^2\right) \\ &\quad = \sum_{p=-\infty}^{\infty} \frac{(-1)^{|p|}}{|p|!}(b)^{(p+|p|)/2}(c)^{(-p+|p|)/2}\,{}_0F_1[-; |p|+1; bc] J_{m+p}(x; z/y; z') t^p, \\ &\quad m \in \mathbb{C}, \ \left|\frac{2bt}{xz}\right| < 1, \ \left|\frac{2c}{xt}\right| < 1. \end{aligned} \tag{3.12}$$

Further if $bc \neq 0$, we can introduce the co-ordinates r and ν such that $b = \frac{r\nu}{2}$ and $c = -\frac{r}{2\nu}$, with these new co-ordinates the matrix elements (3.11) can be expressed as

$$A_{lk}(g) = e^{(m_0+k)a}(-\nu)^{l-k} J_{l-k}(r), \quad k = 0, \pm 1, \pm 2, \ldots, \tag{3.13}$$

and generating relation (3.12) becomes

$$\left(\frac{1+\frac{r}{\nu xt}}{1+\frac{r\nu t}{xz}}\right)^{m/2} J_m\left(x\left(1+\frac{r\nu t}{xz}\right)^{1/2}\left(1+\frac{r}{\nu xt}\right)^{1/2}; z/y\left(1+\frac{r\nu t}{xz}\right)\left(1+\frac{r}{\nu xt}\right)^{-1};\right.$$
$$\left. z'\left(1+\frac{r\nu t}{xz}\right)^{-2}\left(1+\frac{r}{\nu xt}\right)^{2}\right)$$
$$= \sum_{p=-\infty}^{\infty} (-\nu)^p J_p(r) J_{m+p}(x; z/y; z') t^p, \quad \left|\frac{r}{\nu xt}\right| < 1, \ \left|\frac{r\nu t}{xz}\right| < 1, \tag{3.14}$$

which for $t = 1$ reduces to a generalization of Graf's addition theorem [9, p. 44].

4. Applications

We discuss some applications of the generating relations obtained in the preceding section.

(I) Taking $c = 0$ and $t = 1$ in generating relation (3.12), we get

$$\left(1+\frac{2b}{xz}\right)^{-m/2} J_m\left(x\left(1+\frac{2b}{xz}\right)^{1/2}; z/y\left(1+\frac{2b}{xz}\right); z'\left(1+\frac{2b}{xz}\right)^{-2}\right)$$
$$= \sum_{p=0}^{\infty} \frac{(-b)^p}{p!} J_{m+p}(x; z/y; z'), \quad \left|\frac{2b}{xz}\right| < 1. \tag{4.1}$$

Again, taking $b = 0$ and $t = 1$ in generating relation (3.12), we get

$$\left(1-\frac{2c}{x}\right)^{m/2} J_m\left(x\left(1-\frac{2c}{x}\right)^{1/2}; z/y\left(1-\frac{2c}{x}\right)^{-1}; z'\left(1-\frac{2c}{x}\right)^{2}\right)$$
$$= \sum_{p=0}^{\infty} \frac{(-c)^p}{p!} J_{m-p}(x; z/y; z'), \quad \left|\frac{2c}{x}\right| < 1. \tag{4.2}$$

Taking $y = 0$ and $z = 1$ in generating relations (4.1) and (4.2) and using Eq. (2.16), we obtain the formulas of Lommel [16, p. 62, (3.30), (3.31)], respectively. Also taking $y = 0$ and $z = 1$ in generating relation (3.12), and using Eq. (2.16), we obtain [16, p. 62, (3.29)]. Similarly, taking $y = 0$ and $z = t = 1$ in generating relation (3.14), we obtain a generalization of Graf's addition theorem [16, p. 63, (3.32)].

Further, taking $z = z' = 0$ in generating relation (4.2), and using Eq. (2.13), we get

$$\left(1-\frac{2c}{x}\right)^{m/2} H_m\left(\frac{x}{2}\left(1-\frac{2c}{x}\right)^{1/2}, \frac{y}{2}\left(1-\frac{2c}{x}\right)^{-1}\right)$$
$$= \sum_{p=0}^{\infty} \binom{m}{p} (-c)^p H_{m-p}\left(\frac{x}{2}, \frac{y}{2}\right), \quad \left|\frac{2c}{x}\right| < 1, \tag{4.3}$$

where $H_n(x, y)$ is given by Eq. (2.6).

(II) Taking $b = -c$ and $z = z' = t = 1$ in generating relation (3.12), and using Eq. (2.7), we get

$$J_m\left(x\left(1-\frac{2c}{x}\right), y\right)$$
$$= \sum_{p=-\infty}^{\infty} \frac{(-1)^{(3|p|+p)/2}(c)^{|p|}}{|p|!} {}_0F_1\left[-; |p|+1; -c^2\right] J_{m+p}(x, y), \quad \left|\frac{2c}{x}\right| < 1, \tag{4.4}$$

where $J_m(x,y)$ is given by Eq. (2.8). Similarly, taking $z=z'=\nu=t=1$ in generating relation (3.14), we get

$$J_m\left(x\left(1+\frac{r}{x}\right),y\right)=\sum_{p=-\infty}^{\infty}(-1)^p J_p(r)J_{m+p}(x,y),\qquad \left|\frac{r}{x}\right|<1. \tag{4.5}$$

(III) Taking $b=c$, $z=z'=-1$ and $t=1$ in generating relation (3.12), and using Eq. (2.9), we get

$$I_m\left(x\left(1-\frac{2c}{x}\right),y\right)=\sum_{p=-\infty}^{\infty}\frac{(-c)^{|p|}}{|p|!}{}_0F_1\left[-;|p|+1;c^2\right]I_{m+p}(x,y),\qquad \left|\frac{2c}{x}\right|<1, \tag{4.6}$$

where $I_m(x,y)$ is given by Eq. (2.10).

5. Concluding remarks

We note that expressions (3.9) are valid only for group elements g in a sufficiently small neighbourhood of the identity element of the Lie group T_3. However, we can also use operators (3.4) to derive generating relations for 2V2PBF and related functions associated with group elements bounded away from the identity.

If $f(x,y,t;z,z')$ is a solution of the equation $C_{0,0}f=\omega^2 f$, i.e.,

$$\left(-\frac{1}{z}\frac{\partial^2}{\partial x^2}+\frac{4y^2}{x^2z}\frac{\partial^2}{\partial y^2}+\frac{16z'^2}{x^2z}\frac{\partial^2}{\partial z'^2}-\frac{16yz'}{x^2z}\frac{\partial^2}{\partial y\partial z'}-\frac{1}{xz}\frac{\partial}{\partial x}-\frac{4(m-1)y}{x^2z}\frac{\partial}{\partial y}\right.$$
$$\left.+\frac{8(m+2)z'}{x^2z}\frac{\partial}{\partial z'}+\frac{m^2}{x^2z}\right)f(x,y,t;z,z')=\omega^2 f(x,y,t;z,z'), \tag{5.1}$$

then the function $T(g)f$ given by (3.8) satisfies the equation

$$C_{0,0}\big(T(g)f\big)=\omega^2\big(T(g)f\big).$$

This follows from the fact that $C_{0,0}$ commutes with the operators $J^{+\prime}$, $J^{-\prime}$ and $J^{3\prime}$. Now if f is a solution of the equation

$$\left(x_1J^{+\prime}+x_2J^{-\prime}+x_3J^{3\prime}\right)f(x,y,t;z,z')=\lambda f(x,y,t;z,z'), \tag{5.2}$$

for constants x_1,x_2,x_3 and λ, then $T(g)f$ is a solution of the equation

$$\left[T(g)\left(x_1J^{+\prime}+x_2J^{-\prime}+x_3J^{3\prime}\right)T\left(g^{-1}\right)\right]\left[T(g)f\right]=\lambda\left[T(g)f\right]. \tag{5.3}$$

The inner automorphism μ_g of Lie group T_3 defined by

$$\mu_g(h)=ghg^{-1},\qquad h\in T_3, \tag{5.4}$$

induces an automorphism $\mu_g^\star$ of Lie algebra $\mathcal{T}_3$ where

$$\mu_g^\star(\alpha)=g\alpha g^{-1},\qquad \alpha\in\mathcal{T}_3.$$

If $\alpha=x_1\mathcal{J}^++x_2\mathcal{J}^-+x_3\mathcal{J}^3$ where $\mathcal{J}^+,\mathcal{J}^-$ and $\mathcal{J}^3$ are given by Eq. (1.6) and g is given by Eq. (1.5), then we have

$$\mu_g^\star(\alpha)=\left(x_1e^a-bx_3\right)\mathcal{J}^++\left(x_2e^{-a}+cx_3\right)\mathcal{J}^-+x_3\mathcal{J}^3, \tag{5.5}$$

as a consequence of which, we can write

$$\begin{aligned} &T(g)\big(x_1J^{+\prime}+x_2J^{-\prime}+x_3J^{3\prime}\big)T\big(g^{-1}\big)\\ &\quad=\big(x_1e^a-bx_3\big)J^{+\prime}+\big(x_2e^{-a}+cx_3\big)J^{-\prime}+x_3J^{3\prime}. \end{aligned} \tag{5.6}$$

To give an example of the application of these remarks, we consider the function $f(x,y,t;z,z')=J_m(x;z/y;z')t^m$, $m\in\mathbb{C}$. Since $C_{0,0}f=f$ and $J^{3\prime}f=mf$, so the function

$$\begin{aligned} &\big[T(g)f\big](x,y,t;z,z')\\ &\quad=e^{ma}\left(\frac{t^2-\frac{2ct}{x}}{1+\frac{2bt}{xz}}\right)^{m/2}J_m\left(\left(x+\frac{2bt}{z}\right)^{1/2}\left(x-\frac{2c}{t}\right)^{1/2};z/y\left(1+\frac{2bt}{xz}\right)\left(1-\frac{2c}{xt}\right)^{-1};\right.\\ &\qquad\qquad\left.z'\left(1+\frac{2bt}{xz}\right)^{-2}\left(1-\frac{2c}{xt}\right)^{2}\right), \end{aligned} \tag{5.7}$$

satisfies the equations

$$C_{0,0}\big[T(g)f\big]=T(g)f, \tag{5.8}$$

$$\big(-bJ^{+\prime}+cJ^{-\prime}+J^{3\prime}\big)\big[T(g)f\big]=m\big[T(g)f\big]. \tag{5.9}$$

For $a=b=0$ and $c=-1$, we can express function (5.7) in the form

$$h(x,y,t;z,z')=\left(t^2+\frac{2t}{x}\right)^{m/2}J_m\left(\left(x^2+\frac{2x}{t}\right)^{1/2};z/y\left(1+\frac{2}{xt}\right)^{-1};z'\left(1+\frac{2}{xt}\right)^{2}\right). \tag{5.10}$$

Now, using the Laurent expansion

$$h(x,y,t;z,z')=\sum_{k=-\infty}^{\infty}h_k(x,y;z,z')t^k,\quad |xt|<2,$$

in Eq. (5.8), we observe that $h_k(x,y;z,z')$ is a solution of differential equation (2.3) for each integer k. Since the function $h(x,y,t;z,z')$ is bounded for $x=y=0$, therefore we have

$$h_k(x,y;z,z')=c_kJ_k(x;z/y;z'),\quad c_k\in\mathbb{C}.$$

Thus

$$h_k(x,y,t;z,z')=\sum_{k=-\infty}^{\infty}c_kJ_k(x;z/y;z')t^k. \tag{5.11}$$

Now from Eq. (5.9), we have $(-J^{-\prime}+J^{3\prime})h(x,y,t;z,z')=mh(x,y,t;z,z')$ and therefore it follows that

$$c_{k+1}=(m-k)c_k.$$

Further, taking $x=y=0$ in (5.10), and using (5.11), we get $c_0=1/\Gamma(m+1)$, and hence $c_k=1/\Gamma(m-k+1)$. Thus we obtain the following result:

$$\begin{aligned} &\left(t^2+\frac{2t}{x}\right)^{m/2}J_m\left(\left(x^2+\frac{2x}{t}\right)^{1/2};z/y\left(1+\frac{2}{xt}\right)^{-1};z'\left(1+\frac{2}{xt}\right)^{2}\right)\\ &\quad=\sum_{k=-\infty}^{\infty}\frac{J_k(x;z/y;z')t^k}{\Gamma(m-k+1)},\quad |xt|<2, \end{aligned} \tag{5.12}$$

which is not a special case of generating relation (3.12). Several other examples of generating relations can be derived by this method, see for example Weisner [21].

The theory of BF is rich and wide, and certainly provides an inexhaustible field of research. A large number of functions are recognized as belonging to the BF family. Many variable BF were introduced at the beginning of the last century, see for example [1,11], forgotten for many years and reconsidered within the context of various physical applications at the end of the last century, see for example [2–8].

We have considered GBF within the group representation formalism. The 2V2PBF $J_m(x; z/y; z')$ appeared as basis functions for a realization of the representation $Q(-1, m_0)$ of the Lie algebra $\mathcal{T}_3$. The analysis presented in this paper confirms the possibility of extending this approach to other useful forms of GBF.

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